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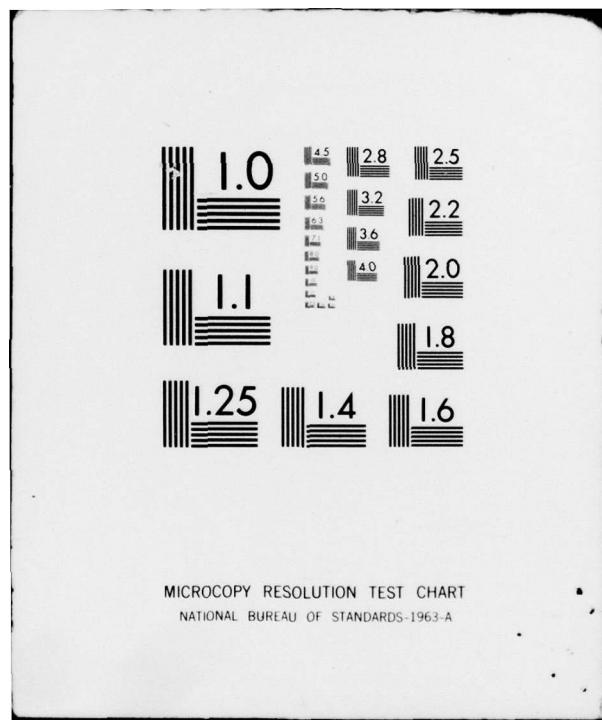
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TOPICS IN NONLINEAR WAVE THEORY

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I. Introduction

The recent developments in nonlinear wave theory over the last ten or fifteen years are already too extensive for a detailed review in one lecture. But it is hoped, through a discussion of some current ideas and their relation to earlier work, to touch upon enough topics to give some general impression of the field.

One of the remarkable advances in the area was the discovery of highly nontrivial exact solutions to some of the prototype equations. These were first found for the Korteweg-de Vries equation

$$\eta_t + \eta\eta_x + \eta_{xxx} = 0. \qquad (1)$$

This equation was derived in 1895 as a long wave approximation for water waves but is of general interest since it combines a simple dispersive linear part, whose dispersion relation is

$$\omega + k^3 = 0, \qquad (2)$$

with a typical nonlinear term. As a consequence it appears as an approximation in other physical settings: plasma physics is an example. Korteweg and de Vries had already noted that solutions of the form $\eta = \eta(x - Ut)$, representing wavetrains translating with constant speed U, could be found in terms of elliptic functions ('cnoidal waves'), and that one limiting case was the solitary wave

$$\eta = 3\alpha^2 \operatorname{sech}^2 \frac{1}{2}(\alpha x - \alpha^3 t). \qquad (3)$$

However, the unexpected boost came from the discovery by Gardner, Greene, Kruskal and Miura (1967) that explicit formulas could be given for the interaction of any number of solitary waves (with different parameters α) and that even more general solutions could be related, via a scattering problem, to the solution of a _linear_ integral equation.

The equation is similar in form to Burgers' equation

$$u_t + uu_x = u_{xx}, \qquad (4)$$

which can be reduced to the heat equation (and hence solved in detail) by the Cole-Hopf transformation

$$u = -2(\log \varphi)_x = -\frac{2\varphi_x}{\varphi}. \qquad (5)$$

But the methods developed required a whole sequence of quite ingenious steps, not just a single trick.

Perhaps the easiest way to show the surprising novelty of this work is to quote immediately the result for N interacting solitary waves. If the parameters are $\alpha_1, \ldots, \alpha_N$, the solution is given by

$$\eta = 12 \frac{\partial^2}{\partial x^2} \log |D|, \qquad (6)$$

where $|D|$ is the determinant with elements

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$$D_{mn} = \delta_{mn} + \frac{2\gamma_m}{\alpha_m + \alpha_n} e^{-\alpha_m x + \alpha_m^3 t}. \quad (7)$$

Imagine calculating second derivatives of determinants, and then even three more derivatives for η_{xxx}, if a direct verification were necessary!

The result of the interaction is also surprising. It can be shown from (6) that the original solitary waves eventually emerge unchanged, and the only memory of the interaction is a constant displacement of position from the path each one would have otherwise followed. Figs. 1 and 2 are for other equations discussed later, but the behavior is typical.

After considerable further development, with many people making contributions, similar results have now been found for other prototypes that combine simple dispersion with typical nonlinear terms. These include

Modified K.dV.: $u_t + u^2 u_x + u_{xxx} = 0$, (8)

Sine-Gordon: $u_{tt} - u_{xx} + \sin u = 0$, (9)

Cubic Schrödinger: $iu_t + u_{xx} + \nu|u|^2 u = 0$, (10)

Boussinesq: $u_{tt} - u_{xx} - (u^2)_{xx} - u_{xxxx} = 0$, (11)

and certain difference equations. It is significant that these were all posed as key equations of the subject, with realistic applications in mind, before the possibility of such general solutions was realized. They were not manufactured to be solvable.

2. Successive approximations and perturbation exparsions.

One of the quickest approaches to results such as (6) starts in a relatively

straightforward way from a perturbation procedure. Since it has not appeared before it will be described briefly. The original motivation was to learn how some of the more standard perturbation procedures, which have been so successful in other nonlinear problems, would fare on these particular equations.

In the case of the Korteweg-de Vries equation (1) it is convenient, but not essential, to introduce

$$\eta = -12\psi_x \quad (12)$$

and take an equivalent form

$$\psi_t + \psi_{xxx} = 6\psi_x^2. \quad (13)$$

Then one would take the series

$$\psi = \sum_1^\infty \psi_j,$$

with or without an explicit small amplitude parameter, and obtain the hierarchy of successive equations

$$\frac{\partial \psi_1}{\partial t} + \frac{\partial^3 \psi_1}{\partial x^3} = 0, \quad (14)$$

$$\frac{\partial \psi_j}{\partial t} + \frac{\partial^3 \psi_j}{\partial x^3} =$$

$$6\{\frac{\partial \psi_1}{\partial x}\frac{\partial \psi_{j-1}}{\partial x} + \frac{\partial \psi_2}{\partial x}\frac{\partial \psi_{j-2}}{\partial x} + \ldots + \frac{\partial \psi_{j-1}}{\partial x}\frac{\partial \psi_1}{\partial x}\}. \quad (15)$$

The simplest solution of (14) is

$$B(x, t) = \gamma e^{-\alpha x + \alpha^3 t}, \quad (16)$$

and this can be used as a preliminary trial case. After ψ_2 and ψ_3 are calculated, it becomes apparent and is readily checked that

$$\psi_j = \frac{(-1)^{j-1} B^j}{\alpha^{j-1}}. \quad (17)$$

The original series is convergent only for large enough positive x; however it is

easily summed to the form

$$\psi = \frac{B}{1+\frac{B}{\alpha}} \qquad (18)$$

which is valid everywhere. It gives in fact the solitary wave (3). Notice that in the final result there is no limitation to small amplitude.

Of course finding the solitary wave is no great achievement, but we see the nature of the problem. For solutions which tend to zero as $x \to \infty$, we might start with a simple perturbation expansion, but then its usefulness will depend on summing the series in some way to obtain a form valid for all x.

This approach has been investigated in detail by Rosales (1977). A general solution of (14) may be written formally as a Fourier integral

$$B(x) = \int e^{i\Omega} d\lambda(k), \qquad \Omega = kx + k^3 t; \qquad (19)$$

we include the formal interpretation that the integral may contain contributions from only discrete points $k = i\alpha_n$ and stands for

$$B(x) = \sum \gamma_n e^{-\alpha_n x + \alpha_n^3 t}. \qquad (20)$$

The t-dependence is not displayed explicitly for reasons that appear in a moment. After ψ_2 and ψ_3 have been calculated it becomes clear and may be verified in general that

$$\psi_j(x) =$$
$$(-2i)^{j-1} \int \ldots \int \frac{e^{i(\Omega_1 + \ldots + \Omega_j)}}{(k_1+k_2)(k_2+k_3)\ldots(k_{j-1}+k_j)}$$
$$d\lambda(k_1)\ldots d\lambda(k_j) \qquad (21)$$

$$=$$
$$(-1)^{j-1} \int_x^\infty \ldots \int_x^\infty B(\frac{x+z_1}{2})B(\frac{z_1+z_2}{2})\cdots B(\frac{z_{j-1}+x}{2})$$
$$dz_1 \ldots dz_{j-1}. \qquad (22)$$

This formula replaces the power B^j in (17) by a more general product. If the similarity can be exploited, there is the possibility of summing the series for ψ in analogy with (18).

First in the case of (20) we may write

$$B(\frac{\xi + \eta}{2}) = p^T(\xi)p(\eta), \qquad (23)$$

where p is a column vector with elements

$$p_n = \left(\gamma_n e^{-\alpha_n x + \alpha_n^3 t}\right)^{1/2} \qquad (24)$$

and p^T denotes the transpose i.e. the row vector. Then, (22) may be written as

$$\psi_j(x) = (-1)^{j-1} p^T(x) P^{j-1}(x) p(x), \qquad (25)$$

where $P(x)$ is the matrix

$$P(x) = \int_x^\infty p(z)p^T(z)dz, \qquad (26)$$

with elements

$$P_{mn}(x) = \frac{2}{\alpha_m + \alpha_n} p_m(x)p_n(x). \qquad (27)$$

In (25) we now have ψ_j expressed as a power; this time as a matrix instead of a scalar. We can evaluate the sum:

$$\psi(x) = \sum_1^\infty (-1)^{j-1} p^T P^{j-1} p$$
$$= p^T(I + P)^{-1} p$$
$$= - \text{Trace}\{(I + P)^{-1} \frac{\partial P}{\partial x}\}$$
$$= - \frac{\partial}{\partial x} \log \det|I + P|. \qquad (28)$$

This gives a form equivalent to (6) for η.

When B is not separable in the form (23), the expression in (22) can be related to the $(j - 1)$th power of an operator applied to B. To do this write $B(\frac{x + y}{2})$ as $B(x, y)$ and introduce an operator $\hat{B}$ defined by

$$\hat{B} f(x, y) = \int_x^\infty f(x, z)B(z, y)dz. \qquad (29)$$

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Then, from (22) the solution can be written

$$\psi(x) = K(x, x), \qquad (30)$$

where

$$K(x, y) = \sum_1^\infty (-1)^{j-1} \hat{B}^{j-1} B(x, y). \qquad (31)$$

Formally, we could write this as

$$K(x, y) = (I + \hat{B})^{-1} B(x, y), \qquad (32)$$

but would then have to give a meaning to the operator $(I + \hat{B})^{-1}$. However, we can instead obtain this result in the form

$$(I + \hat{B})K(x, y) = B(x, y) \qquad (33)$$

directly from (31), and interpret it as

$$K(x, y) + \int_x^\infty K(x, z)B(z, y)dz = B(x, y). \qquad (34)$$

This is the Marchenko integral equation which was originally found as the end result of the 'inverse scattering method'.

The approach goes through with appropriate modifications for the other equations noted in (8)-(11). It is also interesting that when it is applied to Burgers' equation (4) it rapidly leads to

$$u = -2\frac{\partial}{\partial x} \sum_{j=1}^\infty \frac{(-1)^{j-1}}{j} B^j, \qquad (35)$$

where B is any solution of the linear heat equation. Since (35) is the logarithmic series (5), follows with

$$\varphi = 1 + B. \qquad (36)$$

The key step is in finding an effective way to sum the series corresponding to (21) for each case. So far, this step relies very strongly on a simple factored form for the denominator in the integrand. The denominator arises from the linear dispersive operator in the problem (the left hand side

of (15) for the K.dV. equation), since in solving for the j-th iterate as a multiple Fourier integral one divides by

$$\omega(k_1) + \ldots + \omega(k_j) - \omega(k_1 + \ldots + k_j), \qquad (37)$$

where $\omega = \omega(k)$ is the linear dispersion relation. Most of this has to cancel with the numerator and add just a simple factor, such as

$$k_{j-1} + k_j \qquad (38)$$

for the K.dV. equation. At present the requirement of a simple factorization has limited the technique to the equations mentioned. Of course, it is not necessary to sum the perturbation series exactly, this would be too much to expect in more complicated cases. It is only necessary to extract the principal part that leaves the remainder uniformly small.

3. <u>Another example of interacting solitary waves.</u>

Looking beyond these special equations, it is natural to ask whether the results are typical for other problems. In particular, an intriguing question is whether solitary waves, if they exist for the problem, always interact cleanly and eventually emerge with their initial identities intact. For a generalized form of the Korteweg-de Vries equation, which we have studied for other reasons, the answer is yes.

The equation is

$$\eta_t + \eta\eta_x + \int_{-\infty}^\infty V(x - \xi)\eta_\xi(\xi, t)d\xi = 0. \qquad (39)$$

It gives the Korteweg-de Vries equation when

$$V(x) = \delta''(x), \qquad (40)$$

but can give any linear dispersion relation by choosing

$$V(x) = \frac{-1}{2\pi} \int_{-\infty}^{\infty} c(\kappa) e^{i\kappa x} d\kappa, \qquad (41)$$

where $c(\kappa)$ is the desired phase velocity. The original interest (Whitham 1967) was to soften the η_{xxx} term to allow the various breaking phenomena observed in water wave.

A particular case studied is

$$V(x) = \frac{\nu}{2} e^{-\nu|x|}, \qquad c(\kappa) = \frac{\nu^2}{\nu^2 + \kappa^2}, \qquad (42)$$

since (39) can then be reduced to the differential equation

$$(\frac{\partial^2}{\partial x^2} - \nu^2)(\eta_t + \eta\eta_x) - \nu^2\eta_x = 0. \qquad (43)$$

This has solitary wave solutions and a single one can be given analytically. But the expression is complicated enough that it seems at present very unlikely that analytic formulas could be found for the interaction of a number of them. Here the solitary wave solution includes a limiting case of maximum height for which the crest has a sharp angle (relevant to the original purpose in water waves).

Dr. Bengt Fornberg investigated the interactions numerically and found in every case that the solitary waves emerged cleanly with the typical phase changes seen in the earlier work. This was true even for the interaction of the wave of maximum height with a smaller one. See Fig. 2.

Other solutions of (43) were also computed and a selection will be shown.

4. The step problem; the well problem.

The interaction of solitary waves has naturally drawn a lot of attention, but there are many other important problems. These

relate to the question of solving integral equations such as (34) when the function B is not separable.

One basic problem is to find the disturbance produced by an initial step function. In gas dynamics this would produce a shock wave. In the dispersive problems considered here, the equations are reversible, there is no dissipation and the solution must have quite a different character. In the setting of plasma dynamics such solutions are sometimes referred to as 'collisionless shocks'.

For the K.dV. equation the fomulation of this problem leads to a fairly complicated B function in (34) and the solution of the integral equation has not yet been found. However, an earlier approximate approach gives the main parts of the solution. Apart from solitary waves, solutions of dispersive wave problems tend naturally to be oscillatory in character, and then a detailed description would obviously be complicated. With this in mind, an approximate treatment for modulated wavetrains was developed (Whitham 1965, 1974), which derives equations for various overall quantities such as local amplitude $a(x,t)$, local wave number $k(x,t)$, and mean disturbance $\bar{\eta}(x,t)$. This theory was applied to various questions concerning the concept of nonlinear group velocity and certain instability problems, but only halfhearted attempts were made on the step problem because of worries about the validity of the averaging procedures near the front of the disturbance. However, Gurevich and Pitaevskii (1974) show that the theory

gives perfectly reasonable results, at least for the K.dV. equation. In that case, for a step carrying an increase in level

$$\eta = \begin{cases} -1, & x > 0, \\ 0, & x < 0, \end{cases} \tag{44}$$

the quantities a, k, $\bar{\eta}$ are all functions of x/t, the similarity form expected for (44), and the disturbance is confined to

$$-2 < \frac{x}{t} < -\frac{1}{3}. \tag{45}$$

(It should be remembered that in the standard form (1), the description is relative to a moving frame of reference so that the ultimate velocities corresponding to (45) are positive). The solution is given by

$$a = s^2, \quad k = \frac{\pi}{6^{1/2} K(s)}, \quad \bar{\eta} = 2\frac{E(s)}{K(s)} - 2 + s^2, \tag{46}$$

where s is the solution of

$$-\frac{2 - s^2}{3} - \frac{2}{3}\frac{s^2(1 - s^2)K(s)}{E(s) - (1 - s^2)K(s)} = \frac{x}{t}. \tag{47}$$

It is shown in Figures 3 and 4. A typical result of direct numerical calculation of the K.dV. equation (by Dr. Fornberg) is shown in Fig. 5.

In the case of a step that decreases the level, i.e.

$$\eta = \begin{cases} 0, & x > 0, \\ -1, & x < 0, \end{cases} \tag{48}$$

the oscillations are negligible and the approximate solution is just

$$\bar{\eta} = \frac{x}{t}, \quad -1 < \frac{x}{t} < 0. \tag{49}$$

This is a simple solution of the equation

$$\eta_t + \eta\eta_x = 0 \tag{50}$$

when the dispersive term η_{xxx} is neglected. A typical result of direct numerical calculation is shown in Fig. 6.

It would be interesting to obtain these results via the integral equation (34); (49) looks deceptively easy!

Apart from their direct interest the above solutions are useful in bluiding up more involved cases, if only qualitatively. One such case is the rectangular well

$$\eta = \begin{cases} 0, & 0 < x, \\ -1, & -\ell < x < 0, \\ 0, & x < -\ell \end{cases} \tag{51}$$

According to (45) and (49), the disturbance would be a combination of the results for the two separate steps until $t = 3\ell/2$, at which time the two parts begin to interact. Afterwards the solution would be predicted to be something like the numerical results shown in Figs. 7 and 8.

In gas dynamics and similar dissipative systems (also for Burgers' equation (4)) the asymptotic behavior would just be the triangular wave shown in Fig. 9. Dissipation would smooth out the shock at the rear slightly but it could often be approximated as a discontinuity. In suitable normalized variables the approximate solution would satisfy

$$\eta_t + \eta\eta_x = 0 \tag{52}$$

and be given by

$$\eta = \begin{cases} \frac{x}{t}, & -s(t) < x < 0 \\ 0, & x < -s(t) \end{cases} \tag{53}$$

where $x = -s(t)$ is the shock position. The determination of $s(t)$ depends on the choice of the correct shock condition. For these familiar cases, it leads to the condition:

$$\text{velocity} = \frac{1}{2} \times \text{strength} \tag{54}$$

i.e.

$$\dot{s} = \frac{1}{2}\frac{s}{t}, \quad s = At^{1/2}, \quad \text{strength} = At^{-1/2}. \tag{55}$$

Since the triangular wave has width $At^{1/2}$ and depth $At^{-1/2}$, it is immediately seen that this corresponds to constant area

$$\int_{-\infty}^{\infty} \eta \, dx = \text{const.} \qquad (56)$$

In these usual cases, η is proportional to the density and this choice of shock condition is derived from conservation of mass.

In the dispersive wave case of Fig. 7, we might try to view the result as a triangular wave for the mean level $\bar{\eta}$, with the oscillatory tail superposed to carry energy away (in place of dissipation). Then we would have the beginnings of a very simple treatment suitable for more complicated problems. A key question would be the 'shock' condition. Conservation of mass would again suggest (54) and (55). However it is still possible for mass as well as energy to be transferred under the oscillatory tail. An alternative suggested by the step solution (45) rewritten for arbitrary step size, is that the front of the oscillatory part moves with

$$\text{velocity} = \tfrac{1}{3} \times \text{strength}. \qquad (57)$$

Then

$$\dot{s} = \frac{1}{3} \frac{s}{t}; \quad s = At^{\frac{1}{3}}, \quad \text{strength} = At^{-\frac{2}{3}}. \qquad (58)$$

It is intriguing that this corresponds to conservation of

$$\int_{-\infty}^{\infty} |\bar{\eta}|^{1/2} dx, \qquad (59)$$

for this quantity has appeared in significant ways in other connections. First of all, it should be noted that for the solitary wave (3)

$$Z = \int_{-\infty}^{\infty} \eta^{1/2} dx = 2\pi\sqrt{3}, \qquad (60)$$

independent of α. Thus it is the same for all solitary waves, whatever their amplitudes. Secondly, an important prediction derived from the scattering problem referred to in Section 1, and not mentioned so far, concerns the number of solitary waves produced by a large initial disturbance $\eta_0(x)$ in the form of a single positive hump (see Whitham 1974). The number is

$$N = \frac{1}{6^{1/2}\pi} \int_{-\infty}^{\infty} \eta_0^{1/2} dx. \qquad (61)$$

It is interesting that

$$NZ = 2^{1/2} \int_{-\infty}^{\infty} \eta_0^{1/2} dx. \qquad (62)$$

If the quantity

$$\int_{-\infty}^{\infty} \eta^{1/2} dx, \qquad (63)$$

which looks like some kind of 'action', were conserved, the resulting solitary waves would carry it all. Somehow they have picked up more before emerging. But the amount is still proportional to the initial input.

The relation of these results with (58)–(59) is still obscure; it is yet one more instance of the cross-relations that hint there is much more than we know.

Dr. Fornberg's numerical calculations indicate that the width of the triangular wave is proportional to $t^{\frac{1}{3}}$ as in (58), but the strength decays more slowly than $t^{-\frac{2}{3}}$, closer to $t^{-\frac{1}{2}}$ in fact. It is also interesting that the results of water wave experiments by Dr. J. Hammack came out with $t^{-\frac{1}{2}}$ for the strength; any viscous corrections would change the power even further away from 2/3.

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The rectangular well problem has been discussed on the basis of the integral equation (54). A first version by Ablowitz and Newell (1973) was incorrect; a new version by Ablowitz and Segur (1977) apparently favors the power laws in (58).

5. Wavetrain instabilities.

The modulation theory for oscillatory solutions is based on the existence of exact solutions that represent periodic wavetrains. In some cases these wavetrains are unstable to the modulations. This means that small perturbations grow, but there is no implication that the further behavior is chaotic. For the simplest problems, the modulation equations for the amplitude $a(x, t)$ and wavenumber $k(x, t)$ are

$$k_t + \omega_x = 0, \tag{64}$$

$$(a^2)_t + (\omega_0' a^2)_x = 0, \tag{65}$$

where the frequency ω is given by

$$\omega = \omega_0(k) + \omega_2(k)a^2 - \frac{1}{2}\omega_0'' \frac{a_{xx}}{a}; \tag{66}$$

$\omega_0(k)$ is the linear dispersion relation and $\omega_2(k)$ is the coefficient of the nonlinear correction. For the uniform wavetrain a and k are constants, and it is easily shown that perturbed solutions in the form

$$\delta a, \quad \delta k \propto e^{i\mu(x - Vt)} \tag{67}$$

have

$$V = \omega_0' \pm (\omega_0'' \omega_2 a^2 + \frac{1}{4}\omega_0''^2 \mu^2)^{1/2}. \tag{68}$$

If $\omega_0'' \omega_2 > 0$, the values of V are real and give nonlinear generalizations of the linear group velocity. If $\omega_0'' \omega_2 < 0$, small modulations will grow when μ is in the range

$$0 < \mu^2 < 4 \left| \frac{\omega_2}{\omega_0''} \right| a^2. \tag{69}$$

The growth rate depends on

$$\mu \left(|\omega_0'' \omega_2| a^2 - \frac{1}{4}\omega_0''^2 \mu^2 \right)^{1/2} \tag{70}$$

and is maximum for

$$\mu^2 = 2 \left| \frac{\omega_2}{\omega_0''} \right| a^2. \tag{71}$$

These results can also be derived from a discussion of 'side band' interactions, following Benjamin's original analysis of deep water waves. This will be indicated for one of the simplest examples:

$$u_t + 3u^2 u_x + u_{xxx} = 0. \tag{72}$$

Solutions of the form

$$u = \frac{1}{2} A_0(t) e^{ikx} + \frac{1}{2} A_1(t) e^{i(k - \mu)x}$$

$$+ \frac{1}{2} A_2(t) e^{i(k + \mu)x}$$

$$+ \text{complex conjugates}, \tag{73}$$

are considered, where the 'side bands' A_1, A_2 are initially small. In the nonlinear interactions, various products feed back on to the original components and when these are included we have

$$\frac{i}{k} \frac{dA_0}{dt} + k A_0 = \{ \frac{3}{4} A_0 A_0^* + \frac{3}{2} A_1 A_1^*$$

$$+ \frac{3}{2} A_2 A_2^* \} A_0 + \frac{3}{2} A_1 A_2 A_0^*, \tag{74}$$

$$\frac{i}{k - \mu} \frac{dA_1}{dt} + (k - \mu)^2 A_1 = \{ \frac{3}{2} A_0 A_0^* + \frac{3}{4} A_1 A_1^*$$

$$+ \frac{3}{2} A_2 A_2^* \} A_1 + \frac{3}{4} A_0^2 A_2^*, \tag{75}$$

with a similar equation for A_2. If these equations are linearized on the basis that

$$A_{1,2} \ll A_0,$$

the results in (69)-(71) are recovered, in this case with

$$\omega_0 = -k^3, \qquad \omega_2 = \frac{3}{4}k. \qquad (76)$$

The growth rate in (70) is

$$3k\mu\left(\frac{a^2}{2} - \mu^2\right)^{1/2}, \qquad (77)$$

and is a maximum when

$$\mu = a/2. \qquad (78)$$

As the side bands grow at the expense of the main wave, the linearized approximation ceases to apply and the full equation (74)-(75) must be used. This is a conservative system and the energy oscillates between the three modes. Eventually the side-bands decay again and return the energy to the main wave. In detail, it can be shown that

$$\frac{|A_0|^2}{k} + \frac{|A_1|^2}{k-\mu} + \frac{|A_2|^2}{k+\mu} = \text{constant}, \qquad (79)$$

$$\frac{|A_1|^2}{k-\mu} - \frac{|A_2|^2}{k+\mu} = \text{constant}, \qquad (80)$$

and an equation for $|A_1|^2$, say, can be obtained which has solutions in elliptic functions. One limiting case is

$$\frac{|A_1|^2}{k-\mu} = \frac{|A_2|^2}{k+\mu} = \frac{a^2 - |A_0|^2}{2k}$$

$$= \frac{16\mu^2(a^2-2\mu^2)/k}{(12\mu^2+a^2)+(16\mu^2-a^2)\cosh 6k\mu\left(\frac{a^2}{2}-\mu^2\right)^{1/2}t}$$

$$(81)$$

This shows the growth of the side bands in accordance with (77), but a maximum is reached and the side bands decay again. The other solutions are a periodic repetition of this type of behavior.

For the sidebands of maximum growth rate, $\mu = a/2$, the maximum value in (81) gives

$$|A_0|^2 = \frac{3}{7}a^2, \qquad |A_1|^2 = \left(1 - \frac{\mu}{k}\right)\frac{2}{7}a^2,$$

$$|A_2|^2 = \left(1 + \frac{\mu}{k}\right)\frac{2}{7}a^2. \qquad (82)$$

The time taken from 10% of maximum to the maximum is

$$T \simeq 2.5/ka^2 \qquad (83)$$

Numerical studies of this problem were made by Fornberg by direct computation on (72) and the results are extremely interesting. The exact periodic wavetrain was introduced initially, and for some time the disturbance just translated in space without change of form, as it should ideally. However, the small numerical errors were eventually sufficient to trigger the bursts shown in Fig. 10. The clean return to the periodic wavetrain (which was repeated as long as the calculations were run) fits the above analysis. To compare in more detail, the various harmonics in the solution were analyzed and a typical case is shown in Fig. 11. In these computations, the problem is formulated to be periodic over a large space interval; as a consequence, the wave numbers present have to be integer multiples of a basic unit. For the case shown in Fig. 10 and 11 this unit is $\pi/64$. The initial wavetrain has $a = 0.2$, $k = 7$ units. According to (78),

$$\mu = 0.1 \simeq 2 \text{ units,}$$

for the most rapidly growing sidebands. In Fig. 11, the strong growth of side bands 5 and 9 agrees qualitatively with this prediction. But other side bands $k \pm 2\mu$, $k \pm 3\mu$, etc., are also produced by nonlinear interactions which were neglected in the three mode equations (74)-(75). For this case the values in (82) are

$$|A_0| = 0.13, \quad |A_1| = 0.09, \quad |A_2| \simeq 0.12$$

The extra drain on $|A_0|$ can be attributed to loss of energy to the other side bands. The time scale (83) is $T = 180$, which checks quite well.

The bursts repeat with more and more sidebands called into play at each burst, and slowly more activity in between. The case in Figs. 10 and 11 was run to $t = 7600$ with ten bursts recorded.

References

Ablowitz, M.J. and Newell, A.C. 1973. J. Math. Phys. 14, 1277.

Ablowitz, M.J. and Segur, H. 1977. To appear.

Gardner, C.S., Greene, J.M., Kruskal, M.D. and Muira, R.M., 1967. Phys. Rev. Letters 19, 1095.

Gurevich, A.V. and Pitaevskii, L.P. 1974. J.E.T.P. 38, 291.

Korteweg, D.J. and de Vries, G. 1895. Phil. Mag. (5), 39, 422.

Rosales, R.R. 1977. Ph.D. thesis. California Institute of Technology.

Whitham, G.B. 1965. J. Fluid Mech. 22, 273.

Whitham, G.B. 1967. Proc. Roy. Soc. A299, 6.

Whitham, G.B. 1974. Linear and nonlinear waves. Wiley-Interscience.

Fig. 1. Interacting solitary waves for eq. (8)
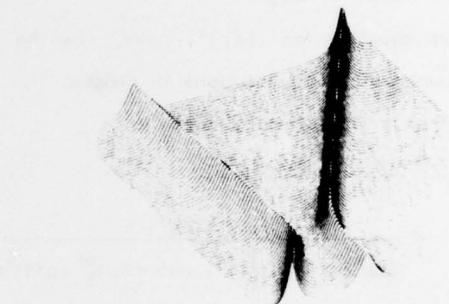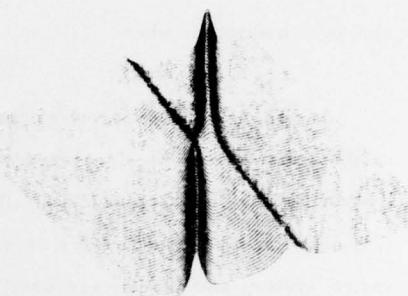
Fig. 2. Interacting solitary waves for eq. (43)
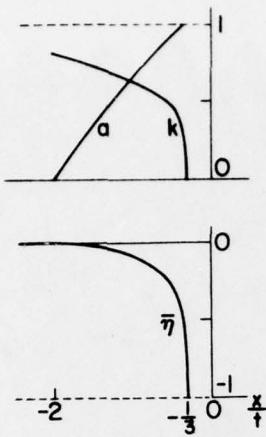
Fig. 3. Amplitude, wave number, mean height for step solution (46).
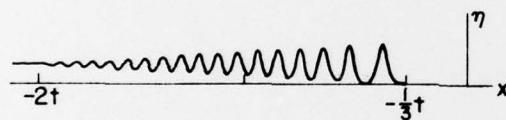
Fig. 4. Wave profile corresponding to Fig. 3.
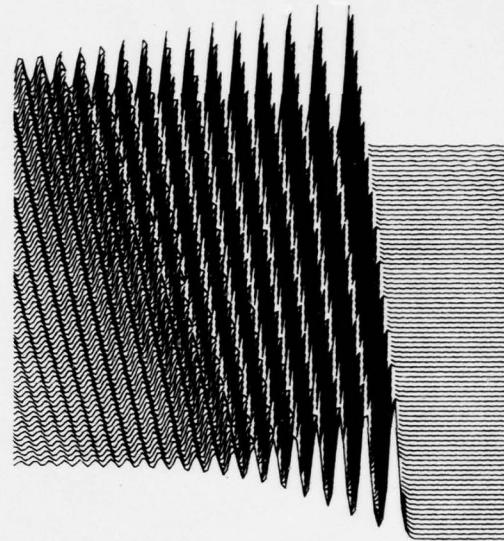
Fig. 5. Numerical calculation of positive step solution.
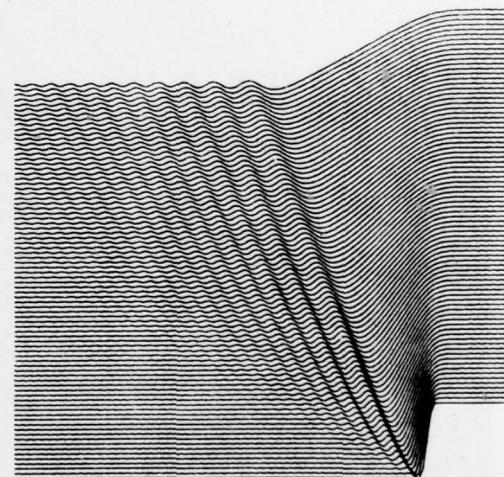
Fig. 6. Numerical calculation of negative step solution.

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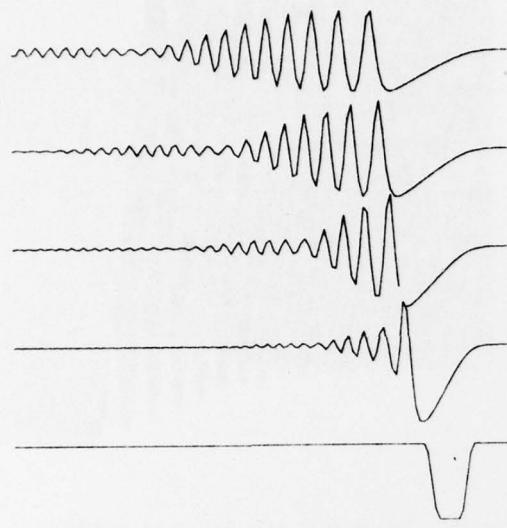
Fig. 7. Numerical calculation for well.
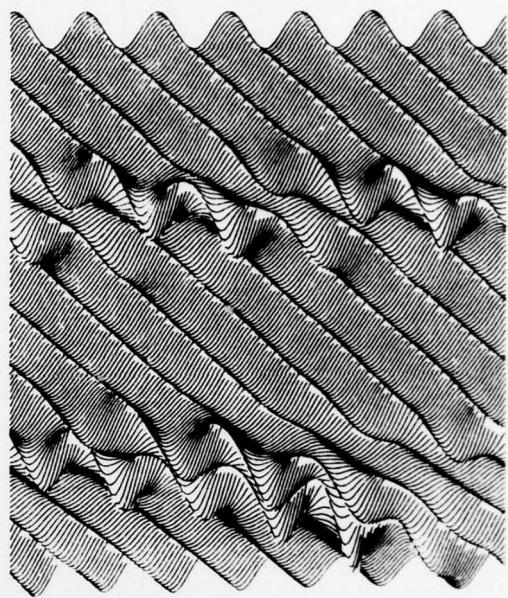
Fig. 10. Instability bursts.
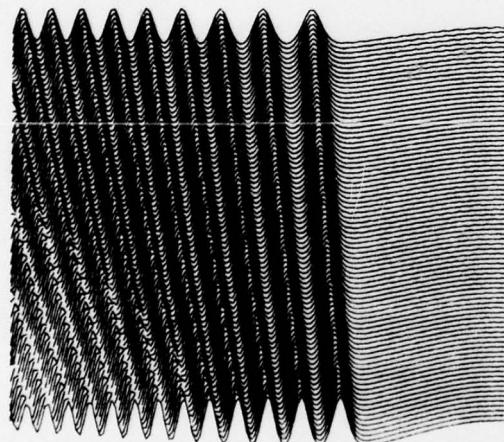
Fig. 8. Numerical calculation for well.
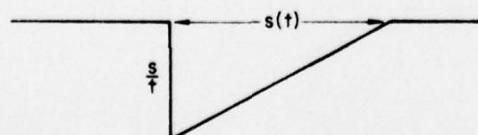
Fig. 9. Simple triangular wave.

Fig. 11. Spectra for Fig. 10.